Int J Theor Phys (2010) 49: 2659–2671
DOI 10.1007/s10773-010-0457-4

Nonlinear Effect and Atomic Entanglement in a Coupled-Cavity Array

H.D. Liu · X.L. Huang · X.X. Yi

Received: 19 April 2010 / Accepted: 28 July 2010 / Published online: 18 August 2010

Abstract By introducing the nonlinear effects that arise from Kerr medium, we theoretically study the nonlinear effect and the entanglement between two atoms in two coupled cavities. We give out the process of dynamic stability and solve the eigen problem of the system under high-intensive fields. The dynamics of the two coupled cavity with high-intensity fields inside is also studied numerically, the effects of atom-field coupling on the self-trapping as well as on the entanglement are also analyzed and discussed. In vacuum and high-intensity fields we calculate the concurrence of the two atoms in both theoretical and realistic situation, and discuss the nonlinear effect on the atomic entanglement. The result shows that the nonlinear interaction can play a controlling role in entangling two atoms.

Keywords Cavity-QED · Kerr medium · Coupled-cavity array · Entanglement · Nonlinear effect

1 Introduction

As an important development of quantum electrodynamics (QED), cavity-QED has played a central role in quantum-information processing (QIP) [1, 2]. Photons and atoms are coupled together in the presence of some macroscopic bodies, like mirrors (metallic or dielectric), cavity walls or waveguides, which can provide good separation between the system and environment. One of the applications of cavity QED in QIP is coupled cavity arrays (CCAs), in which quantum information can distributed through the system by the photons located in each cavity. Compared with the usual waveguide, CCAs have several interesting potential applications, such as QIP and simulations of quantum strongly correlated many-body

H.D. Liu · X.L. Huang · X.X. Yi (✉)
School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, China
e-mail: yixx@dlut.edu.cn

X.L. Huang
School of Physics and Electronic Technology, Liaoning Normal University, Dalian 116029, China

systems. Recently, considerable theoretical investigations have been carried out to implement quantum state transfer [1, 3–5], entanglement generation [3, 6–8], and two-qubit gates [9, 10] by using a CCAs.

Nonlinearity is usually introduced to quantum systems due to mean field treatment of many-body model and/or system-environment couplings. The former has been widely discussed in Bose-Einstein condensates (BECs) which quantum properties have attracted considerable interest since 1995. The quantum dynamics of an atom BEC in a double-well potential was studied in [11]. Micheli et al. [12] have discussed the creation of many-particle entangled states of a two-component BEC. The phenomenon of coherent destruction of tunneling is discussed in [13]. When properly chosen initial conditions, [11–15] predicted and studied an interesting feature known as self-trapping (introduced by Landau [16]). This feature can be observed in quantum optics system with Kerr medium [17]. Recently the Kerr medium has been implemented in movable micro-mirror [18] and nanomechanical resonators [19], and its effect on the evolution of Wigner function of quantum states was discussed in [20].

In the present paper, we shall examine the entanglement between two two-level atoms contained in a coupled-cavity array. Nonlinear effects arising from Kerr medium in cavities are taken into account. We calculate the concurrence of the two atoms in both vacuum and high-intensity fields, and thus discuss the nonlinear effect on the atomic entanglement, which shows different fields of atomic entanglement compared with the linear system. The paper is organized as follows. In Sect. 2 we present a general formulism for the coupled-cavity array. By the static analysis, the dynamic stability process and dressed states under high intensive fields are given in Sect. 3. In Sect. 4, we give a numerical simulation for the dynamics of the coupled-cavity array system with high-intensity fields to analyze the nonlinear effect. In both theoretical and realistic situation, the atomic entanglement is calculated and discussed in vacuum and high-intensity fields case in Sect. 5. Finally, a summary of results is presented and discussed in Sect. 6.

2 Formalism

The setup we shall study is introduced in Fig. 1, where we consider the system consists of two coupled cavities with frequency ω, each containing a two-level system with Rabi frequency Ω. The interaction between atom and cavity is described by the Jaynes-Cummings model (under the Rotating Wave Approximation). Photons can tunnel between the two cavities with a rate A. We ignore the cavity decay and atomic spontaneous emission (which will be discussed in Sect. 5). The linear Hamiltonian therefore can be written as

$$H^l = \sum_{j=1}^{2} \Big[\Omega|e\rangle_j \langle e| + \omega a_j^\dagger a_j + J(a_j\sigma_j^+ + a_j^\dagger \sigma_j^-) \Big] + A(a_1^\dagger a_2 + a_2^\dagger a_1), \tag{1}$$

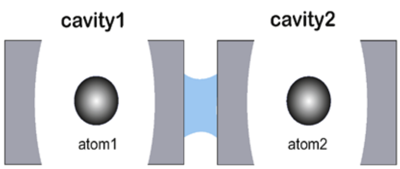

Fig. 1 (Color online) Schematic illustration of the coupled-cavity system,which consists of two coupled cavities. Each cavity contains a two-level atom. Photon hopping can occur between the cavities

where $a_j^\dagger$ and a_j are the creation and annihilation operators of the jth cavity respectively. σ_j^+ and σ_j^- are the atomic spin flip operators for the jth cavity. $|e\rangle_j$ and $|g\rangle_j$ represent the ground and excited states of the atom in the jth cavity respectively. J describes the atom-cavity coupling strength, and A is the hopping strength between cavities. Compared with previous works [3], we introduce the influence of nonlinearity on the transparent properties of photons in the coupled-cavity.

In nonlinear optics, the Kerr medium can be described by expanding the polarization as a power series in the field amplitude as

$$\mathbf{P} = \chi^{(1)} \cdot \mathbf{E} + \chi^{(2)} : \mathbf{EE} + \chi^{(3)} \vdots \mathbf{EEE} + \cdots, \tag{2}$$

where $\chi^{(n)}$ is a $(n+1)$th rank susceptibility tensor and $\cdot$ denotes the tensor product. For single-mode electric field, if we expanding the series to third order and neglect the term in $\chi^{(2)}$ for lacking phase matching, the Hamiltonian of the field becomes [21]

$$H = : \int d^3\mathbf{r} \left\{ \frac{|\mathbf{B}|^2}{2\mu_0} + \mathbf{E}\left[\frac{1}{2}(\epsilon_0 + \chi^{(\mathbf{1})})\mathbf{E} + \frac{1}{4}\chi^{(\mathbf{3})}\mathbf{EEE} \right] \right\} :, \tag{3}$$

where : : denotes normal ordering. If we quantize the field, the Hamiltonian in the form of second quantization can be written as [22, 23]

$$H = \omega a^\dagger a + g(a_j^\dagger a_j)^2, \tag{4}$$

where ω is the frequency of the field, and the coupling to the Kerr medium modeled by an anharmonic oscillator of strength g which determined by $\chi^{(3)}$. Hence, taking nonlinearity into account, the linear Hamiltonian in (1) becomes

$$H = H^l + g \sum_{j=1}^{2} (a_j^\dagger a_j)^2. \tag{5}$$

In order to investigate the effect of the nonlinearity introduced in (5), we also consider the situation with very high field intensities where the classical part of quantities are dominant. We can therefore treat physical quantities as classical value and quantum fluctuations, namely decompose every operator as its average value plus a small fluctuation. In our model, this yields

$$\begin{aligned} a_j &= \alpha_j + \delta a_j, \\ \sigma^{+,-,z} &= \langle \sigma^{+,-,z} \rangle + \delta\sigma^{+,-,z}, \end{aligned} \tag{6}$$

where $\alpha_j = \langle a_j \rangle$ and $j = 1, 2$. Substituting (6) into the Heisenberg equation $i\hbar \frac{\partial F}{\partial t} = [F, H]$, we obtain the following equations for average values,

$$\begin{aligned} i\hbar\dot{\alpha}_1 &= \omega\alpha_1 + J\langle \sigma_1^- \rangle + A\alpha_2 + 2g|\alpha_1|^2\alpha_1, \\ i\hbar\dot{\alpha}_2 &= \omega\alpha_2 + J\langle \sigma_2^- \rangle + A\alpha_1 + 2g|\alpha_2|^2\alpha_2, \\ i\hbar\langle \dot{\sigma}_j^- \rangle &= \Omega\langle \sigma_j^- \rangle - J\alpha_j\langle \sigma_j^z \rangle, \\ i\hbar\langle \dot{\sigma}_j^z \rangle &= -2J\alpha_j^*\langle \sigma_j^- \rangle + 2J\alpha_j\langle \sigma_j^+ \rangle, \\ i\hbar\langle \dot{\sigma}_j^+ \rangle &= -\Omega\langle \sigma_j^+ \rangle + J\alpha_j^*\langle \sigma_j^z \rangle, \end{aligned} \tag{7}$$

and the equations for fluctuations,

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\delta a_1 &= \omega\delta a_1 + J\delta\sigma_1^- + A\delta a_2 + 4g|\alpha_1|^2\delta a_1 + 2g\alpha_1^2\delta a_1^\dagger, \\
i\hbar\frac{\partial}{\partial t}\delta a_2 &= \omega\delta a_2 + J\delta\sigma_2^- + A\delta a_1 + 4g|\alpha_2|^2\delta a_2 + 2g\alpha_2^2\delta a_2^\dagger, \\
i\hbar\frac{\partial}{\partial t}\delta\sigma_j^- &= \Omega\delta\sigma_j^- - J\alpha_j\delta\sigma_j^z - J\delta a_j\langle\sigma_j^z\rangle, \\
i\hbar\frac{\partial}{\partial t}\delta\sigma_j^z &= -2J(\delta a_j^\dagger\langle\sigma_j^-\rangle - \alpha_j\delta\sigma_j^+ + \alpha_j^*\delta\sigma_j^- - \delta a_j\langle\sigma_j^+\rangle).
\end{aligned}
\tag{8}
$$

We find that the Hamiltonian in (5) and the motion equations (7) are familiar to us. If we remove the atoms from the two cavities, the Hamiltonian will be

$$H = \sum_{j=1}^{2}\left[\omega a_j^\dagger a_j + g(a_j^\dagger a_j)^2\right] + A(a_1^\dagger a_2 + a_2^\dagger a_1) \tag{9}$$

which takes the same form as the Hamiltonian describing the BECs in a double-well potential [11], and the motion equations

$$i\hbar\dot{\alpha}_j = \omega\alpha_1 + A\alpha_{3-j} + 2g|\alpha_j|^2\alpha_j \quad (j = 1, 2) \tag{10}$$

are just the discrete self-trapping equation (DST) [24] of the double-well potential system. With the atoms involved, (7) can also show some of the same features of the DST equation and some different effects caused by the atoms, which we will discussed in Sect. 4.

In deriving (7) and (8), we have separated the contributions of average value from those of fluctuations. The dynamical stability of the system can be obtained by (8), we will present the details in next section.

3 Static Analysis

To analyze the linear stability of the system, we first consider the steady-state solutions of (7) which can be obtained by setting the left hand of (7) to zero. By $\langle\vec{\sigma}_j\rangle^2 = 4|\langle\sigma_j^-\rangle|^2 + |\langle\sigma_j^z\rangle|^2 = 1$, we get

$$
\begin{aligned}
\langle\sigma_j^z\rangle &= \frac{\pm\Omega}{\sqrt{\Omega^2 + 4J^2|\alpha_j|^2}}, \\
\langle\sigma_j^-\rangle &= \frac{\pm J\alpha_j}{\sqrt{\Omega^2 + 4J^2|\alpha_j|^2}}.
\end{aligned}
\tag{11}
$$

Substituting (11) into the steady-state equation, we find a cubic equation for $I = |\alpha|^2$

$$
\begin{aligned}
&16g^2J^2I^3 + 4g[g\Omega^2 + 4J^2(\omega - A)]I^2 + 4(\omega - A)[g\Omega^2 + J^2(\omega - A)]I \\
&\quad + (\omega - A)^2\Omega^2 - J^4 = 0,
\end{aligned}
\tag{12}
$$

where we set $|\alpha_1|^2 = |\alpha_2|^2 = |\alpha|^2 \neq 0$ and $\alpha_1 = -\alpha_2 = \alpha$ by using the symmetry of the equations. The steady-state solution given by the real root of (12) can be obtained as

$$I = \frac{\{B + \frac{[g\Omega^2 - 2J^2(\omega - A)]^2}{B} - [g\Omega^2 + 4J^2(\omega - A)]\}}{12gJ^2}, \tag{13}$$

where

$$B = (\sqrt{[2J^2(\omega - A) - g\Omega^2]^3 + 27gJ^8} + 3\sqrt{3g}J^4)^{2/3}. \tag{14}$$

Thus, we can get all the steady-state solutions by substituting it into (11). With the steady-state solutions, we can linearize the equation of motion by using (8) which can be written in matrix form as

$$d\delta\vec{V} = M\delta\vec{V}dt, \tag{15}$$

where $\delta\vec{V} = [\delta a_1, \delta a_1^\dagger, \delta a_2, \delta a_2^\dagger, \delta\sigma_{1-}, \delta\sigma_{1+}, \delta\sigma_{2-}, \delta\sigma_{2+}, \delta\sigma_{1z}, \delta\sigma_{2z}]^T$, and M is

$$M = \begin{pmatrix} -i(\omega + 4g|\alpha|^2) & -2ig\alpha^2 & -iA & 0 & -iJ & 0 & 0 & 0 & 0 & 0 \\ 2ig\alpha^{*2} & i(\omega + 4g|\alpha|^2) & 0 & iA & 0 & iJ & 0 & 0 & 0 & 0 \\ -iA & 0 & -i(\omega + 4g|\alpha|^2) & -2ig\alpha^2 & 0 & 0 & -iJ & 0 & 0 & 0 \\ 0 & iA & 2ig\alpha^{*2} & i(\omega + 4g|\alpha|^2) & 0 & 0 & 0 & iJ & 0 & 0 \\ iJ\langle\sigma_{1z}\rangle & 0 & 0 & 0 & -i\Omega & 0 & 0 & 0 & iJ\alpha & 0 \\ 0 & -iJ\langle\sigma_{1z}\rangle & 0 & 0 & 0 & i\Omega & 0 & 0 & -iJ\alpha^* & 0 \\ 0 & 0 & iJ\langle\sigma_{2z}\rangle & 0 & 0 & 0 & -i\Omega & 0 & 0 & -iJ\alpha \\ 0 & 0 & 0 & -iJ\langle\sigma_{2z}\rangle & 0 & 0 & 0 & i\Omega & 0 & iJ\alpha^* \\ -2iJ\langle\sigma_{1+}\rangle & 2iJ\langle\sigma_{1-}\rangle & 0 & 0 & 2iJ\alpha^* & -2iJ\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -2iJ\langle\sigma_{2+}\rangle & 2iJ\langle\sigma_{2-}\rangle & 0 & 0 & -2iJ\alpha^* & 2iJ\alpha & 0 & 0 \end{pmatrix}, \tag{16}$$

where we have taken $\hbar = 1$. The linear stability depends on whether the eigenvalues of the matrix M having non-positive real part. Although the solutions of the eigenvalues are rather complicated, we can still find some properties of the eigenvalues λ:

$$\lambda_{1,2} = 0, \qquad \lambda_3 = -\lambda_4, \qquad \lambda_5 = -\lambda_6, \qquad \lambda_7 = -\lambda_8, \qquad \lambda_9 = -\lambda_{10}, \tag{17}$$

and if we take the high coupling limit (J is very large), the eigenvalues can be simplified to

$$\lambda_{1,2} = 0,$$
$$\lambda_{3,4} = \pm\sqrt{-2|\alpha|^2 R_+ - 2|\alpha|\sqrt{|\alpha|^2 R_-^2 - 6gJ^2}},$$

$$\lambda_{5,6} = \pm\sqrt{-2|\alpha|^2 R_+ + 2|\alpha|\sqrt{|\alpha|^2 R_-^2 - 6gJ^2}},$$
$$\lambda_{7,8} = \pm\sqrt{2|\alpha|^2 R_- - 2|\alpha|\sqrt{|\alpha|^2 R_+^2 - 6gJ^2}}, \tag{18}$$
$$\lambda_{9,10} = \pm\sqrt{2|\alpha|^2 R_- + 2|\alpha|\sqrt{|\alpha|^2 R_+^2 - 6gJ^2}},$$

where $R_\pm = J^2 \pm 3g^2\alpha^2$. From the discussion above, we find that the eigenvalues of M are five pairs of contrary number, so if λ are not all zero, at least one eigenvalue will have positive real part. It means the steady-state solutions are unstable except $|\alpha| = \langle\sigma_j^-\rangle = 0$, $\langle\sigma_j^z\rangle = 1$.

Next we turn to the eigen problem of the Hamiltonian in (5). In strong fields limitation, we can treat the fields as classic quantities. Thus the Hamiltonian can be approximated to

$$H = \sum_{j=1}^{2}\left[\Omega|e\rangle_j\langle e| + J(\alpha_j\sigma_{j-} + \alpha_j^*\sigma_{j+})\right] + E_0, \tag{19}$$

where

$$E_0 = \sum_{j=1}^{2}\left(\omega|\alpha_j|^2 + g|\alpha_j|^4\right) + A(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1). \tag{20}$$

Therefore the atomic states $\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\}$ can form a complete basis, and we can give out the eigenvalues of the Hamiltonian and the eigenstates in the form of dressed states (unnormalized)

$$\begin{aligned}
E_1 &= -\frac{1}{2}(F_1 + F_2) + E_0, & \phi_1 &= \begin{bmatrix} -\frac{F_1}{2J\alpha_1} \\ \frac{F_1(F_2+2\Omega)}{4J^2\alpha_1\alpha_2^*} \\ 1 \\ -\frac{F_2+2\Omega}{2J\alpha_2^*} \end{bmatrix}, \\
E_2 &= \Omega + \frac{1}{2}(F_1 - F_2) + E_0, & \phi_2 &= \begin{bmatrix} \frac{F_1+2\Omega}{2J\alpha_1} \\ -\frac{(F_1+2\Omega)(F_2+2\Omega)}{4J^2\alpha_1\alpha_2^*} \\ 1 \\ -\frac{F_2}{2J\alpha_2^*} \end{bmatrix}, \\
E_3 &= \Omega + \frac{1}{2}(F_2 - F_1) + E_0, & \phi_3 &= \begin{bmatrix} -\frac{F_1}{2J\alpha_1} \\ -\frac{F_1F_2}{4J^2\alpha_1\alpha_2^*} \\ 1 \\ \frac{F_2}{2J\alpha_2^*} \end{bmatrix}, \\
E_4 &= 2\Omega + \frac{1}{2}(F_1 + F_2) + E_0, & \phi_4 &= \begin{bmatrix} \frac{F_1+2\Omega}{2J\alpha_1} \\ \frac{(F_1+2\Omega)F_2}{4J^2\alpha_1\alpha_2^*} \\ 1 \\ \frac{F_2}{2J\alpha_2^*} \end{bmatrix},
\end{aligned} \tag{21}$$

where

$$F_j = \sqrt{\Omega^2 + 4J^2|\alpha_j|^2} - \Omega \quad (j = 1, 2). \tag{22}$$

We can find that the eigenenergy can be separated into two parts: one is the non-interaction energy of the atoms and fields, the other is the interaction energy determined by F_1 and F_2. Without the atom-field interaction ($J = 0$), E_2 and E_3 will degenerate to one energy level, and the energy levels will share same span Ω. So in case of $|\alpha_1|^2 \neq |\alpha_2|^2$, the interaction energy can "break" the degenerate of E_2 and E_3. But if the two fields have same intensities: $|\alpha_1|^2 = |\alpha_2|^2$, the eigenstates of the system will degenerate again. In this case, the degenerate energy level will no longer contain interaction part, and the energy level span will be enlarged to $\Omega + F_1 + F_2$. Correspondingly, the nonlinear terms cannot influence spans between energy levels and eigenvectors in strong field situation, but the nonlinear energy plays a dominant role in eigenenergy for high field intensities.

4 Dynamics

To get further insight on the nonlinear effect on the exchange of photons between the two cavities, we calculate the dynamics of the system by numerical solution of (7). In this aspect, we first notice the conservation of the total excitation number $N_T = \sum_{j=1}^{2}(|e\rangle_j\langle e| + a_j^\dagger a_j)$. We can thus normalize the atoms and photons by using the relation $|e\rangle_j\langle e| + |g\rangle_j\langle g| = 1$ and $N_{photon} = N_T - \sum_{j=1}^{2}|e\rangle_j\langle e|$ respectively, where N_{photon} is the total photon number in the two cavities, we decompose σ_z into $\sigma_e \equiv |e\rangle\langle e|$ and $\sigma_g \equiv |g\rangle\langle g|$ by $\langle\dot{\sigma}_e\rangle = -\langle\dot{\sigma}_g\rangle = \langle\dot{\sigma}_z\rangle/2$. This leads to a new set of coupled equations from (7). We plot the time evolution of photon numbers in the two cavities in Figs. 2 and 3. The initial state is chosen to be $|e\rangle_1|g\rangle_2|10, 0\rangle$. Figure 2 shows the effect of the nonlinear coupling on the photon number $|\alpha|^2$ in each cavity with $g = 0, 0.205\omega, 0.21\omega$ and $\omega/2$, respectively. From Figs. 2(b) and (c) we can find that, when g exceed a certain value, the fluctuation ranges of the photon numbers diminish significantly, leading to the self-trapping for photons. If g is large enough the fluctuation of photon number will tend to be zero as shown in Fig. 2(d). It means the nonlinear effect not

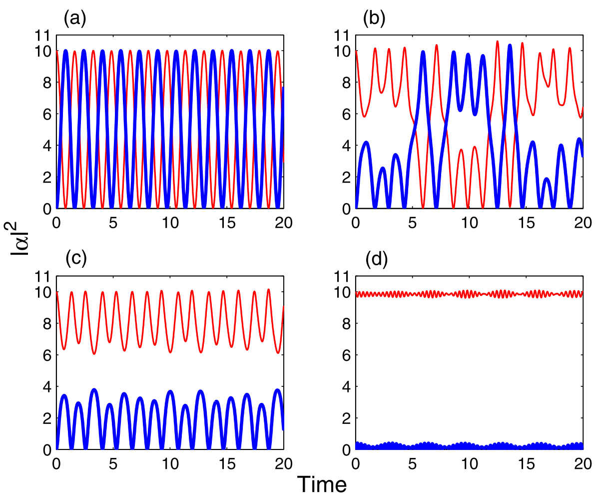

Fig. 2 (Color online) Time evolution of photon number in 1th cavity (*red* and *thin line*) and 2th cavity (*blue* and *think line*). The parameters are $N_m = 10$, $N_n = 0$, $\Omega = \omega$, $A = \omega$, $J = \omega/2$. (**a**) $g = 0$, (**b**) $g = 0.205\omega$, (**c**) $g = 0.21\omega$, and (**d**) $g = \omega/2$

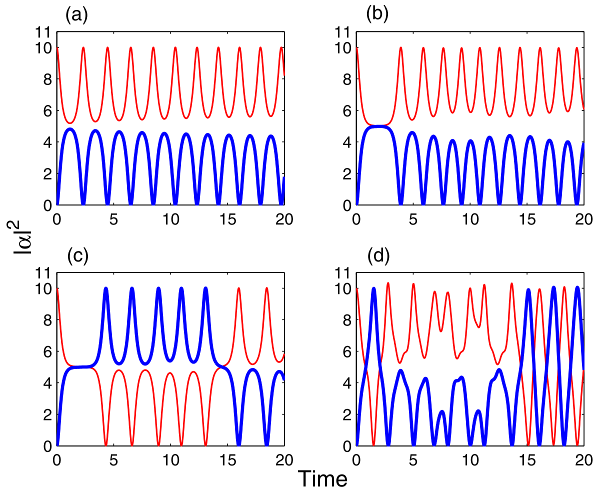

Fig. 3 (Color online) The same dynamics of photon number as Fig. 2 with the parameters $N_m = 10$, $N_n = 0$, $\Omega = \omega$, $A = \omega$, $g = 0.20\omega$. (**a**) $J = 0$, (**b**) $J = 0.0668\omega$, (**c**) $J = 0.0669\omega$, and (**d**) $J = \omega/2$

only block the exchange between the two cavities, but also "froze" the photon number in each cavity. Figure 3 shows that for different atom-field interaction strengthes, the critical point of self-trapping shifts. Figure 3(a) illustrates the photon numbers without atom-field interaction, the Hamiltonian in (5) has the same for as the Hamiltonian of BECs in a double-well potential [15], which yields a critical point for self-trapping $g = 2A/N_{photon} = 0.2\omega$. The self-trapping of photons occurs. From Figs. 3(b), (c) and (d), we find the atom-field interaction depress the self-trapping of photons, and therefore shifts its critical point. It can be explained that the photon-atom coupling makes an influence on the self-trapping.

From the dynamics discussed above, we can find these phenomena: with high field intensities, when the nonlinear coupling strength g reach a critical point, the fluctuations of the photon numbers of the two cavities suddenly decreased, this is the self-trapping for photons. As g continues to increases, the photon number fluctuations become smaller and smaller. When g is large enough, the photon numbers of the two cavities remains unchanged. Since both of the atom-field and field-field interaction are blocked in high g limit due to photon self-trapping, the entanglement between the two atoms will be conserved which we will present in next section. We also found the atom-field interaction can influence the photon self-tapping, the stronger the coupling is, the larger the nonlinear interaction strength is required for self-trapping to occur.

5 Entanglement

Now, we investigate the nonlinear effect on the entanglement between the two atoms. First we consider the situation that each field is initially in the vacuum state. To study the entanglement between the two atoms we choose the initial state of the entire system as

$$|\psi(0)\rangle = (\cos\theta|e\rangle_1|g\rangle_2 + \sin\theta|g\rangle_1|e\rangle_2)|00\rangle, \tag{23}$$

where $|00\rangle$ is the vacuum states of the two fields, which can be prepared by the similar method in [25, 26], and the atomic state $\cos\theta|e\rangle_1|g\rangle_2 + \sin\theta|g\rangle_1|e\rangle_2$ is an arbitrary entangled states of the two atoms determined by different θ. We can previously couple the two atoms through a high quality single-mode cavity to produce a two-qubit gate $U(\theta)$ [27] and apply this gate directly to the untangled state $|eg\rangle$ to prepare such a state, then put the two atoms into the two cavities respectively. Since the system is restricted to one-excitation subspace, its state at time t can thus be written as

$$
\begin{aligned}
|\psi_1(t)\rangle = {} & c_1(t)|g\rangle_1|g\rangle_2|10\rangle + c_2(t)|g\rangle_1|g\rangle_2|01\rangle \\
& + c_3(t)|e\rangle_1|g\rangle_2|00\rangle + c_4(t)|g\rangle_1|e\rangle_2|00\rangle.
\end{aligned}
\tag{24}
$$

For the initial state given in (23), the coefficients $c_1(0) = c_2(0) = 0, c_3(0) = \cos\theta$ and $c_4(0) = \sin\theta$. By introducing delocalized modes

$$
\begin{aligned}
C_j(t) &\equiv c_1(t) + (-1)^j c_2(t), \\
D_j(t) &\equiv c_3(t) + (-1)^j c_4(t), \quad (j = 1, 2),
\end{aligned}
\tag{25}
$$

and substituting them into the Schrödinger equation, (24) can be solved exactly

$$
\begin{aligned}
C_j(t) &= e^{-i(\Omega-\Lambda_j/2)t}\left[C_j(0)\cos(\kappa_j t) + \frac{i}{\kappa_j}\left(\frac{\Lambda_j}{2}C_j(0) - JD_j(0)\right)\sin(\kappa_j t)\right], \\
D_j(t) &= e^{-i(\Omega-\Lambda_j/2)t}\left[D_j(0)\cos(\kappa_j t) - \frac{i}{\kappa_j}\left(\frac{\Lambda_j}{2}D_j(0) + JC_j(0)\right)\sin(\kappa_j t)\right],
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
\kappa_j &= \sqrt{\left(\frac{\Lambda_j}{2}\right)^2 + J^2}, \\
\Lambda_j &= \Delta - (-1)^j A - g, \qquad \Delta = \Omega - \omega.
\end{aligned}
\tag{27}
$$

The total density operator is denoted by $\rho_T = |\psi\rangle\langle\psi|$. The reduced density operator for the atoms can be obtained by tracing out the field modes

$$
\rho_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |c_3|^2 & c_3c_4^* & 0 \\ 0 & c_4c_3^* & |c_4|^2 & 0 \\ 0 & 0 & 0 & |c_1|^2 + |c_2|^2 \end{pmatrix}.
\tag{28}
$$

The concurrence [28] of the atoms is thus given by

$$
C(t) = 2|c_3(t)|\cdot|c_4(t)| = \frac{1}{2}|D_2^2(t) - D_1^2(t)|,
\tag{29}
$$

C is a complicated function of time, however, when we consider the large nonlinear limit $(g \gg \{J, \Delta, A\})$, (29) can be simplified to

$$
C(t) = \sqrt{1 - \cos^2 2\theta \cos^2(\kappa_1 - \kappa_2 + A)t},
\tag{30}
$$

which denotes the concurrence is oscillating from $C(0) = |\sin(2\theta)|$ to 1 with a frequency $2(\kappa_1 + \kappa_2 + A)$.

For large nonlinear coupling strength , the concurrence C evolves like an Absolute sine function in area $[C(0), 1]$ with a frequency $2(\kappa_1 - \kappa_2 + A)$. This means without populating the fields, the maximum entanglement with different initial states can always generated at time $\pi/[2(\kappa_1 - \kappa_2 + A)]$, and keep the entanglement no smaller than that at the initial time. Furthermore, it is possible to make a perfect state transfer in high g limit, as suggested in [3]. By (26), for an initial state $|e\rangle_1|g\rangle_2|00\rangle$, when $g \gg \{J, \Delta, A\}$, the perfect state transfer will occur at times $T = n\pi/(\kappa_2 - \kappa_1 + A)$, $(n \in N)$.

From (26) and (27), we can find that, the nonlinear coupling strength g acts as a new detuning, the large g limit is equal to the large detuning limit, the nonlinear effect is not obvious.

Next we consider the high field intensities situation, we will find the nonlinear effect plays a notable role on the evolution of the concurrence. The initial state of the system is set to be

$$|\psi(0)\rangle = (\cos\theta|e\rangle_1|g\rangle_2 + \sin\theta|g\rangle_1|e\rangle_2)|N_m N_n\rangle, \tag{31}$$

where $|N_m N_n\rangle$ is the Fock state of the two fields and the total excitation number $N_T = N_m + N_n + 1$ is conserved. The state of the system at time t is thus given as

$$|\psi(t)\rangle = \sum_{ijmn} c_{ijmn}|ijmn\rangle, \tag{32}$$

where $i, j = 0, 1$, $|0\rangle \equiv |g\rangle$, $|1\rangle \equiv |e\rangle$, m, n is the photon number of the cavity 1 and 2 respectively, and $|mn\rangle$ is the Fock state of the system. The sum is under the restriction of $i + j + m + n = N_T$. As discussed above, by solving the Schrödinger equation, we obtain a set of differential equations for the coefficient c_{ijmn}

$$\begin{aligned} i\hbar\dot{c}_{ijmn} &= \left[\Omega(i+j) + \omega(m+n) + g(m^2+n^2)\right]c_{ijmn} \\ &\quad + A\left[\sqrt{(m+1)n}\,c_{ijm+1n-1} + \sqrt{m(n+1)}\,c_{ijm-1n+1}\right] \\ &\quad + J\Big[i\sqrt{m+1}\,c_{i-1jm+1n} + j\sqrt{n+1}\,c_{ij-1mn+1} \\ &\quad + (1-i)\sqrt{m}\,c_{i+1jm-1n} + (1-j)\sqrt{n}\,c_{ij+1mn-1}\Big]. \end{aligned} \tag{33}$$

The density matrix for the atoms can be written as

$$\rho_a = \sum_{i'j'ij} d_{i'j'ij}|i'j'\rangle\langle ij| = \begin{pmatrix} d_{0000} & 0 & 0 & 0 \\ 0 & d_{0101} & d_{0110} & 0 \\ 0 & d^*_{0110} & d_{1010} & 0 \\ 0 & 0 & 0 & d_{1111} \end{pmatrix}, \tag{34}$$

where

$$d_{i'j'ij} = \sum_{mn} c_{i'j'mn}c^*_{ijmn}, \tag{35}$$

the sum is under the restriction of $i + j + m + n = i' + j' + m + n = N_T$. From the equations above, we have performed extensive numerical calculations for the concurrence C, and present the result in Fig. 4.

Figure 4 shows the time evolution of the concurrence with $\theta = \pi/4$. We find the concurrence tend to be a constant, as the nonlinear interaction strength g increase, which is different from the case with low field intensities. With this technique, it is possible to storage the entanglement. The phenomenon is not only for $\theta = \pi/4$. No matter how large the

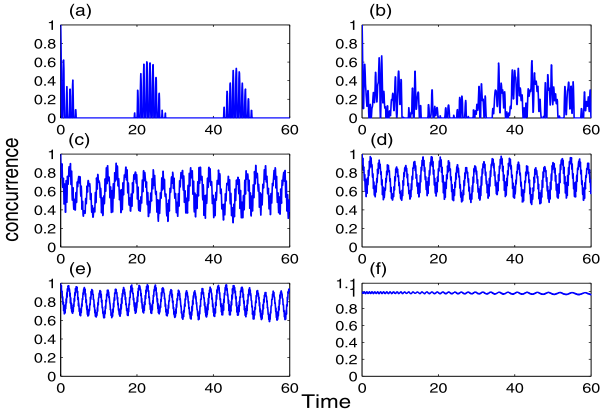

Fig. 4 (Color online) Time dependence of concurrence of the two atoms. The parameters are $N_m = 10$, $N_n = 0$, $\theta = \pi/4$, $\Omega = \omega$, $A = \omega$, $J = \omega/2$. (**a**) $g = 0$, (**b**) $g = \omega/4$, (**c**) $g = \omega/2$, (**d**) $g = 3\omega/4$, (**e**) $g = \omega$, and (**f**) $g = 5\omega$

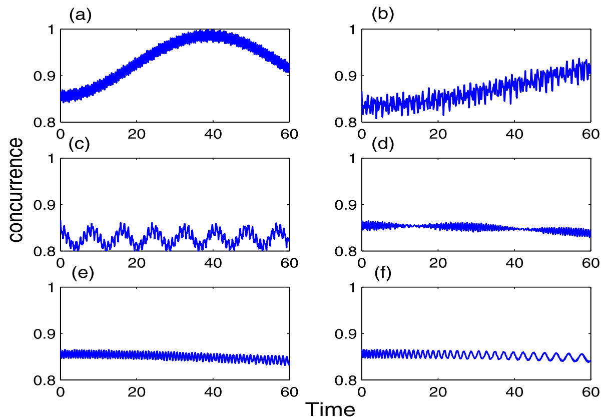

Fig. 5 (Color online) The same dynamics of concurrence as Fig. 4 with different parameters $\theta = \pi/6$, $g = 5\omega$. (**a**) $N_m = 0$, (**b**) $N_m = 1$, (**c**) $N_m = 2$, (**d**) $N_m = 4$, (**e**) $N_m = 6$, and (**f**) $N_m = 10$

initial entanglement is, the concurrence will always tend to constant with high nonlinear interaction. Thus, the same phenomena will not be presented here.

Figure 5 shows the same dynamics of concurrence as Fig. (4) with $\theta = \pi/6$ and different photon numbers. Figure 5(a) illustrates the concurrence C in vacuum field, and Figs. 5(b), (c), (d), (e) and (f) show the effect of nonlinear coupling on the concurrence with photon number $N_m = 1, 2, 4, 6$ and 10, respectively. We can find that as N_m increasing, the evolution of concurrence tends to a straight line, the nonlinear effect arises.

The phenomena we discussed above can be understood by examining the Hamiltonian in (5), from which we can find that the evolution of entanglement between the two atoms is determined by two parts: one is the interaction between atom and field in each cavity with strength J, the other is the photon exchange between the two cavities with strength A. As we discussed in the previous section, the nonlinear effect can block the photon transfer between the two cavities with strong nonlinearity, much the same as the self-trapping for BECs in a double-well potential. Therefore, the entanglement can be always kept at the same level as initial.

Now we consider a more realistic model in which we take cavity decay and atomic spontaneous emission into consideration. If no photon is detected, the Hamiltonian in (5) can be

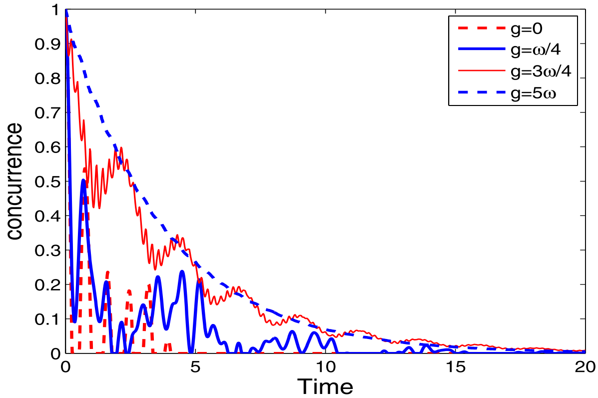

Fig. 6 (Color online) Time dependence of concurrence with cavity decay and atomic spontaneous emission. The parameters are $N_m = 10$, $N_n = 0$, $\theta = \pi/4$, $\Omega = \omega$, $A = \omega$, $J = \omega/2$, $\gamma = 0.005\omega$, $\eta = 0.005\omega$, and g take different values

conditionally written as

$$H' = H^l + g\sum_{j=1}^{2}(a_j^\dagger a_j)^2 - \sum_{j=1}^{2}\left(i\gamma|e\rangle_j\langle e| + i\eta a_j^\dagger a_j\right), \tag{36}$$

where γ and η are the decay rates of atoms and fields, respectively. The concurrence of the two atoms can still be numerically calculated from the density matrix in (29) and (34) by taking the transform: $\Omega \to \Omega - i\gamma$, $\omega \to \omega - i\eta$. From (26), (27) and (29), we can find that the concurrence will decay with time exponentially in the vacuum field situation, and the nonlinear strength has no effect on decay. In the high field intensities situation, we can still calculate the density matrix in (34) numerically, and present the result in Fig. 6.

From Fig. 6 we can see that the time evolution of the concurrence with $\theta = \pi/4$ with cavity decay and atomic spontaneous emission. The red dash line illustrates the situation without nonlinear effect, the concurrence rapidly decay to zero. With the nonlinear effect, we find that as the nonlinear strength g increase, the decay of the concurrence gradually slow down, i.e. the nonlinear effects can depress the entanglement decay.

6 Conclusion

In summary , we have studied theoretically the influence of nonlinearity on the entanglement between two atoms in two coupled cavities. We have made static analysis of the system and gave out the process of dynamic stability analysis. The eigen problem has also been solved. To give an intuitive view, we also numerically simulated the dynamics of the system. We found the following features. (1) When the nonlinear interacting strength g reaches a critical value, the self-trapping phenomenon occurs. (2) In the presence of self-trapping, the larger g is, the smaller the photon numbers fluctuate. (3) The atom-field interaction affects the self-trapping a lot, in particular at the critical point. The concurrence that measures the entanglement was calculated and discussed. We have shown that the nonlinear strength can play a controlling role in entangling the atoms. For cavities with vacuum fields inside, we found the nonlinearity characterized by g plays a role similar to the detuning. It means we can use this mechanism to generate maximum entangled state, and make perfect state transfer between the two atoms without populating the field. When field intensity is very high, the evolution of the concurrence tends to be a constant, indicating that entanglement is conserved in this situation. This phenomenon is due to the blockade of photon transfer between the two cavities.

References

1. Cirac, J.I., Ekert, A.K., Huelga, S.F., Macchiavello, C.: Phys. Rev. A **59**, 4249 (1999)
2. Berman, P.R.: Cavity Quantum Electrodynamics. Academic Press, New York (1994)
3. Ogden, C.D., Irish, E.K., Kim, M.S.: Phys. Rev. A **78**, 063805 (2008)
4. Bose, S., Angelakis, D.G., Burgrth, D.: J. Mod. Opt. **54**, 2307 (2007)
5. Biswas, A., Agarwal, G.S.: Phys. Rev. A **70**, 022323 (2004)
6. Cho, J., Angelakis, D.G., Bose, S.: Phys. Rev. A **78**, 022323 (2008)
7. Irish, E.K.: Phys. Rev. A **80**, 043825 (2009)
8. Giampaolo, S.M., Illuminati, F.: Phys. Rev. A **80**, 050301 (2009)
9. Devitt, S.J., Greentree, A.D., Ionicioiu, R., O'Brien, J.L., Munro, W.J., Hollenberg, L.C.L.: Phys. Rev. A **76**, 052312 (2007)
10. Xiao, Y.F., Gao, J., Zou, X.B., McMillan, J.F., Yang, X., Chen, Y.L., Han, Z.F., Guo, G.C., Wong, C.W.: arXiv:0707.2632
11. Milburn, G.J., Corney, J., Wright, E.M., Walls, D.F.: Phys. Rev. A **55**, 4318 (1997)
12. Micheli, A., Jaksch, D., Cirac, J.I., Zoller, P.: Phys. Rev. A **67**, 013607 (2003)
13. Gong, J.B., Morales-Molina, L., Hänggi, P.: Phys. Rev. Lett. **103**, 133002 (2009)
14. Wu, B., Niu, Q.: Phys. Rev. A **61**, 023402 (2000)
15. Tonel, A.P., Links, J., Foerster, A.: J. Phys. A **38**, 1235 (2008)
16. Landau, L.D.: Phys. Z. Sowjetunion **3**, 664 (1993)
17. Gong, Z.R., Ian, H., Liu, Y.X., Sun, C.P., Nori, F.: Phys. Rev. A **80**, 065801 (2009)
18. Kumar, T., Bhattacherjee, A.B., ManMohan: arXiv:0909.3617
19. Semião, F.L., Furuya, K., Milburn, G.J.: Phys. Rev. A **79**, 063811 (2009)
20. Stobińska, M., Milburn, G.J., Wódkiewicz, K.: Phys. Rev. A **78**, 013810 (2008)
21. Bloembergen, N.: Nonlinear Optics. Benjamin, New York (1965)
22. Drunmond, P.D., Walls, D.F.: J. Phys. A **13**, 725 (1980)
23. Werner, M.J., Risken, H.: Phys. Rev. A **44**, 4623 (1991)
24. Elibeck, J.C., Lomdahl, P.S., Scott, A.C.: Physica D **16**, 318 (1985)
25. Mancini, S., Vitali, D., Tombesi, P.: Phys. Rev. Lett. **80**, 688 (1998)
26. Genes, C., Vitali, D., Tombesi, P., Gigan, S., Aspelmeyer, M.: Phys. Rev. A **77**, 033804 (2008)
27. Zhu, S.L., Wang, Z.D., Zanardi, P.: Phys. Rev. Lett. **94**, 100502 (2005)
28. Wooters, W.K.: Phys. Rev. Lett. **80**, 2245 (1998)